

# Model-Independent Determinations of $\bar{B} \rightarrow D l \bar{\nu}, D^* l \bar{\nu}$ Form Factors

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We present nonperturbative, model-independent parametrizations of the individual QCD form factors relevant to  $\bar{B} \rightarrow D^* l \bar{\nu}$  and  $\bar{B} \rightarrow D l \bar{\nu}$  decays. These results follow from dispersion relations and analyticity, without recourse to heavy quark symmetry. To describe a form factor with two percent accuracy, three parameters are necessary, one of which is its normalization at zero recoil,  $\mathcal{F}(1)$ . We combine the individual form factors using heavy quark symmetry to extract values for the product  $|V_{cb}|\mathcal{F}(1)$  from  $\bar{B} \rightarrow D^* l \bar{\nu}$  data with negligible extrapolation uncertainty.

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## 1. Introduction

A determination of the Cabibbo-Kobayashi-Maskawa element  $|V_{cb}|$  from the decays  $\bar{B} \rightarrow D^* l \bar{\nu}$  and  $\bar{B} \rightarrow D l \bar{\nu}$  requires knowledge of the transition amplitudes  $\langle D^*(p', \epsilon) | (V_\mu - A_\mu) | \bar{B}(p) \rangle$  and  $\langle D(p') | V_\mu | \bar{B}(p) \rangle$ , respectively. In the limit of infinitely heavy  $b$  and  $c$  quark masses these amplitudes are predicted[1–3] at one kinematic point, namely, when the recoiling  $D^*$  or  $D$  is at rest in the rest frame of the decaying  $\bar{B}$ . In terms of  $q \equiv p - p'$ , this zero recoil point occurs at  $q^2 = q_{\text{max}}^2 \equiv (M - m)^2$ , where  $M = m_B$  is the decaying meson mass and  $m = m_D$  or  $m_{D^*}$  is the mass of the final state meson. Unfortunately, the differential decay width  $d\Gamma/dq^2$  vanishes at  $q_{\text{max}}^2$ , so an extraction of  $|V_{cb}|$  requires the extrapolation of the matrix element from  $q^2$  values less than  $q_{\text{max}}^2$ .

This method has been used by several experiments[4–6]. ARGUS[6] tested the importance of the extrapolation on the determination of  $|V_{cb}|$  by using various parametrizations. The observed variations of  $|V_{cb}|$  were larger than the rest of the errors combined. In principle the error inherent in the extrapolation can be made arbitrarily small by collecting an arbitrarily large amount of data, arbitrarily close to  $q_{\text{max}}^2$ ; such an endeavor is impractical, if not unattainable. Therefore, a precision measurement of  $|V_{cb}|$  from  $\bar{B} \rightarrow D l \bar{\nu}$  or  $D^* l \bar{\nu}$  requires a model-independent understanding of the extrapolation.

In a previous letter[7] we presented such a model-independent extrapolation. To this effect we used analyticity, crossing symmetry, and QCD dispersion relations to find a two-parameter fit to the  $B$ -meson  $b$ -number elastic form factor  $F(q^2)$ . Heavy quark symmetries were then invoked to relate  $F(q^2)$  to the amplitude for  $\bar{B} \rightarrow D^* l \bar{\nu}$ ; given the validity of heavy quark symmetries, we showed that over the relevant range of  $q^2$  the accuracy of the two-parameter fit was better than 1%.

With a two-parameter extrapolation at hand, experimentalists can accurately determine  $|V_{cb}|$  by making a simultaneous fit of the data to  $|V_{cb}|$  and the two parameters in our extrapolation. At the moment this program suffers from two main theoretical uncertainties:

- 1 ) Incalculable nonperturbative corrections to the amplitudes for  $\bar{B} \rightarrow D l \bar{\nu}$  and  $D^* l \bar{\nu}$  at  $q_{\text{max}}^2$  appear at orders  $1/m_c$  and  $1/m_c^2$ , respectively[8]. The size of these is controversial.
- 2 ) The extrapolation of Ref. [7] relies on heavy quark spin and flavor symmetries, with *a priori* corrections of order  $1/m_c$ .

Of these issues only the latter is addressed in this paper. Instead of assuming particular numerical values for the normalization of form factors at zero recoil and making a fit to

two parameters plus  $|V_{cb}|$ , we evade the first issue by presenting our results as three-parameter fits in units of  $\mathcal{F}(1)$ , the amplitude at  $q_{\text{max}}^2$ . We then improve on the method of Ref. [7] by dropping the unnecessary use of heavy quark symmetries. To this end we derive, in Sec. 2, bounds on the form factors describing the amplitudes for  $\bar{B} \rightarrow Dl\bar{\nu}$  and  $D^*l\bar{\nu}$ . As in Ref. [7], our arguments are based on QCD dispersion relations, crossing symmetry, and analyticity. The bounds take the form of integrals over the unphysical region  $q^2 > (M+m)^2$ , which are then related to the individual form factors in the physical region  $0 \leq q^2 \leq q_{\text{max}}^2$ . In Sec. 3 we derive our parametrizations by constructing quantities from each form factor that can be legitimately expressed as Taylor series with bounded coefficients. These parametrizations constitute our main results. Some technical issues are addressed in Sec. 4, where we demonstrate that the error incurred by ignoring cuts in the form factors is negligible. Section 5 enumerates the corrections to the parametrization and estimates their effects on the bounds. We discuss the sensitivity of our method to such corrections and demonstrate that their effects are minimal, thus establishing the robustness of the technique. In Sec. 6 we use heavy quark symmetry to relate the separate form factors appearing in the measured rate, but point out that measurements of individual form factors in the near future will obviate the need for this use of heavy quark symmetries. We present results from fits of current data to this parametrization, including values for  $|V_{cb}|\mathcal{F}(1)$ , in Sec. 7. Our conclusions appear in Sec. 8.

## 2. Dispersion Relations

The QCD matrix elements governing the semileptonic decays  $\bar{B} \rightarrow D^*l\bar{\nu}$  and  $\bar{B} \rightarrow Dl\bar{\nu}$  may be expressed in terms of the form factors

$$\begin{aligned}\langle D^*(p', \epsilon) | V^\mu | \bar{B}(p) \rangle &= ig\epsilon^{\mu\alpha\beta\gamma}\epsilon_\alpha^* p'_\beta p_\gamma \\ \langle D^*(p', \epsilon) | A^\mu | \bar{B}(p) \rangle &= f_0\epsilon^{*\mu} + (\epsilon^* \cdot p)[a_+(p+p')^\mu + a_-(p-p')^\mu] \\ \langle D(p') | V^\mu | \bar{B}(p) \rangle &= f_+(p+p')^\mu + f_-(p-p')^\mu\end{aligned}\tag{2.1}$$

where  $V^\mu = \bar{c}\gamma^\mu b$ , and  $A^\mu = \bar{c}\gamma^\mu\gamma_5 b$ . In terms of these form factors, the differential decay widths for  $\bar{B} \rightarrow Dl\bar{\nu}$  and  $\bar{B} \rightarrow D^*l\bar{\nu}$  are respectively

$$\frac{d\Gamma}{dq^2} = \frac{|V_{cb}|^2 G_F^2 (k^2 q^2)^{\frac{3}{2}}}{24\pi^3 M^3} |f_+|^2\tag{2.2}$$

and

$$\frac{d\Gamma}{dq^2} = \frac{|V_{cb}|^2 G_F^2 \sqrt{k^2 q^2}}{96\pi^3 M^3} [2q^2 |f_0|^2 + |F_1|^2 + 2q^4 k^2 |g|^2], \quad (2.3)$$

where

$$F_1 = \frac{1}{m} \left[ 2q^2 k^2 a_+ - \frac{1}{2}(q^2 - M^2 + m^2) f_0 \right] \quad (2.4)$$

determines the partial width to longitudinally polarized  $D^*$ 's, and  $f_0$  and  $g$  respectively determine the axial and vector contributions from transversely polarized  $D^*$ 's (longitudinal polarizations do not contribute to the vector matrix element in the  $\bar{B}$  rest frame, as is readily seen from Eq. (2.1)).  $k^2$  is related to the three-momentum squared  $\mathbf{p}_D^2$  for  $D$  or  $D^*$  in the  $\bar{B}$  rest frame, and is given by

$$k^2 = \frac{M^2}{q^2} \mathbf{p}_D^2 = \frac{1}{4q^2} [q^2 - (M + m)^2][q^2 - (M - m)^2], \quad (2.5)$$

with  $M$  and  $m$  the  $\bar{B}$  and  $D$  or  $D^*$  meson masses, respectively.

In our derivation of constraints from dispersion relations, we follow the well-known methods developed by authors listed in Ref. [9]. We begin by considering the two-point function

$$\Pi_J^{\mu\nu}(q) = (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_J^T(q^2) + g^{\mu\nu} \Pi_J^L(q^2) \equiv i \int d^4x e^{iqx} \langle 0 | T J^\mu(x) J^{\dagger\nu}(0) | 0 \rangle, \quad (2.6)$$

where  $J = V$  or  $A$ . In QCD we can render both sides of this relation finite by making one subtraction. We thus obtain the once-subtracted dispersion relations

$$\chi_J^{T,L}(q^2) \equiv \frac{\partial \Pi_J^{T,L}}{\partial q^2} = \frac{1}{\pi} \int_0^\infty dt \frac{\text{Im} \Pi_J^{T,L}(t)}{(t - q^2)^2}. \quad (2.7)$$

The functions  $\chi_J^{T,L}(q^2)$  may be computed reliably in perturbative QCD for values of  $q^2$  far from the kinematic region where the current  $J$  can create resonances: specifically,  $(m_b + m_c) \Lambda_{\text{QCD}} \ll (m_b + m_c)^2 - q^2$ . For resonances containing a heavy quark, it is sufficient to take  $q^2 = 0$ .

The absorptive part  $\text{Im} \Pi_J^{\mu\nu}(q^2)$  is obtained by inserting on-shell states between the two currents on the right-hand side of Eq. (2.6). For  $\mu = \nu$ , this is a sum of positive-definite terms, so one can obtain strict inequalities by concentrating on the term with intermediate

states of  $B$ - $D$  or  $B$ - $D^*$  pairs. The contribution of  $B$ - $D^*$  pairs to the right-hand side of (2.7) enters (no sum on  $\mu$ ) as

$$\begin{aligned} \text{Im } \Pi_J^{\mu\mu}(t = q^2) &\geq \frac{n_f}{2} \int d\Omega \frac{\sqrt{k^2}}{16\pi^2 \sqrt{q^2}} \theta(q^2 - (M + m)^2) \\ &\sum_{\epsilon} \langle 0 | J^{\dagger\mu} | B(\frac{q}{2} - k) D^*(\frac{q}{2} + k, \epsilon) \rangle \langle B(\frac{q}{2} - k) D^*(\frac{q}{2} + k, \epsilon) | J^\mu | 0 \rangle, \end{aligned} \quad (2.8)$$

with an analogous form (no sum over polarizations) for  $B$ - $D$  pairs. Here  $n_f$  is the number of light valence quark flavors for the  $B$  and  $D$  or  $D^*$  that give physically equivalent contributions; in practice, we take  $n_f = 2$ . The momentum  $q$  here and subsequently is not to be confused with  $q$  in Eq. (2.7), which will subsequently be set to zero. The matrix elements in Eq. (2.8) are related by crossing symmetry to those in Eq. (2.1). That is, they are described by the same form factors, but defined in different regions of the complex  $q^2$  plane.  $k^2$  is still defined by Eq. (2.5) but may now be interpreted as the three-momentum squared of either the  $B$  or  $D, D^*$  in the center of mass frame. For massless leptons it turns out that the partial widths appearing in Eq. (2.3) present the same combinations of form factors as the space-space components of Eq. (2.8). It therefore suffices to use the dispersion relation

$$\chi_J = \frac{1}{\pi} \int_0^\infty dt \frac{\text{Im } \Pi_J^{ii}(t)}{t^3}, \quad (2.9)$$

where  $\chi_J = \chi_J^T(0) - \frac{1}{2} \frac{\partial}{\partial q^2} \chi_J^L(0)$ . This definition of  $\chi_J$  corresponds to the combination of  $\Pi_J^T$  and  $\Pi_J^L$  that gives  $\Pi_J^{ii}$  at  $q^2 = 0$ . At one loop,

$$\begin{aligned} \chi_V(u) = \chi_A(-u) &= \frac{1}{32\pi^2 m_b^2 (1 - u^2)^5} \\ &\times [(1 - u^2)(3 + 4u - 21u^2 + 40u^3 - 21u^4 + 4u^5 + 3u^6) \\ &+ 12u^3(2 - 3u + 2u^2) \ln u^2], \end{aligned} \quad (2.10)$$

where  $u = \frac{m_c}{m_b}$  is the ratio of quark masses. For  $u = 0.33$ ,  $\chi_V = 9.6 \cdot 10^{-3}/m_b^2$  and  $\chi_A = 5.7 \cdot 10^{-3}/m_b^2$ .

When substituted into Eq. (2.9), (2.8) and (2.10) lead to bounds on integrals of the analytically continued form factors. For example, for the axial current  $J = A$ , Eq. (2.8) becomes

$$\text{Im } \Pi_A^{ii} \geq \frac{n_f \sqrt{k^2}}{12\pi \sqrt{q^2}} \left[ |f_0|^2 + \frac{1}{2q^2} |F_1|^2 \right] \theta(q^2 - (M + m)^2). \quad (2.11)$$

The bound in this case, which may be taken to constrain  $|f_0|$  and  $|F_1|$  separately, reads

$$\frac{n_f}{12\pi^2\chi_A} \int_{(M+m)^2}^{\infty} dq^2 \frac{\sqrt{k^2}}{(q^2)^{\frac{7}{2}}} \left[ |f_0|^2 + \frac{1}{2q^2} |F_1|^2 \right] \leq 1. \quad (2.12)$$

We now define a new variable  $z$  by

$$\frac{1+z}{1-z} = \sqrt{\frac{(M+m)^2 - q^2}{4Mm}}. \quad (2.13)$$

Taking the principal branch of the square root in this expression, the change of variables  $q^2 \rightarrow z$  maps the two sides of the cut  $q^2 > (M+m)^2$  to the unit circle  $|z| = 1$ , with the rest of the  $q^2$  plane mapped to the interior of the unit circle. In particular, the real values  $-\infty < q^2 \leq (M-m)^2$  and  $(M-m)^2 \leq q^2 < (M+m)^2$  are mapped to the real axis,  $1 > z \geq 0$  and  $0 \geq z > -1$  respectively. Written in terms of  $z$ , the inequalities from Eqs. (2.8)-(2.10) now read

$$\frac{1}{2\pi i} \int_C \frac{dz}{z} |\phi_i(z) F_i(z)|^2 \leq 1. \quad (2.14)$$

The contour  $C$  is the unit circle. The weighing functions are

$$\phi_i = M^{2-s} 2^{2+p} \sqrt{\kappa n_f} [r(1+z)]^{\frac{p+1}{2}} (1-z)^{s-\frac{3}{2}} [(1-z)(1+r) + 2\sqrt{r}(1+z)]^{-s-p}, \quad (2.15)$$

where  $r = m/M$  is the ratio of meson masses, and  $\kappa$ ,  $p$  and  $s$  depend on the form factors  $F_i$  as listed in Table 1.

$i$	$F_i$	$1/\kappa$	$p$	$s$
0	$f_0$	$12\pi M^2 \chi_A$	1	3
1	$F_1$	$24\pi M^2 \chi_A$	1	4
2	$g$	$12\pi M^2 \chi_V$	3	1
3	$f_+$	$6\pi M^2 \chi_V$	3	2

Table 1. Factors entering Eq. (2.15) for the form factors  $F_i$ .

The results (2.14)-(2.15) apply equally well to analogous heavy-to-light form factors such as in  $\bar{B} \rightarrow K^* \gamma$  and  $\bar{B} \rightarrow \pi l \nu$ ; for the latter process, they agree with Ref. [10] upon substitution of  $m_\pi$  for  $m_D$ .

### 3. Parametrization of Form Factors

Our parametrizations of the form factors rely on a Taylor expansion about  $z = 0$ . To connect this expansion to bounds at  $|z| = 1$ , we need a function which is analytic inside the unit disk. The form factors  $F_i$  have cuts and poles along the segment  $q^2 > (M - m)^2$  of the real axis in the complex  $q^2$  plane, and therefore only on the segment  $(-1, 0)$  of the real axis in  $z$  or on the unit circle  $|z| = 1$ .

We have used the freedom to redefine  $\phi_i$  by a phase to ensure that it has no poles, branch cuts, or zeros in the interior of the unit circle  $|z| < 1$ , but the form factors  $F_i(q^2)$  have poles due to the existence of stable spin-one states with unit bottom and charm number (spin-zero states only contribute to  $f_-$  and  $a_-$ , which, for massless leptons, give vanishing contribution to the differential rate). The masses of these  $B_c^*$  mesons can be reliably computed[11–13] with potential models. The vector states are predicted to have masses corresponding (for  $z$  defined with  $m = m_{D^*}$ ) to  $z_1 = -0.284$ ,  $z_2 = -0.472$ ,  $z_3 = -0.531$ , and  $z_4 = -0.907$ , while the axial vector masses correspond to  $z_5 = -0.395$ ,  $z_6 = -0.399$ ,  $z_7 = -0.609$ , and  $z_8 = -0.619$ . One may form functions  $P(z)$  that are products of terms of the form  $(z - z_i)/(1 - \bar{z}_i z)$ , known to mathematicians as Blaschke factors[14]:

$$\begin{aligned} P_0 = P_1 &= \prod_{j=5}^8 \frac{(z - z_j)}{(1 - \bar{z}_j z)}, \\ P_2 = P_3 &= \prod_{j=1}^4 \frac{(z - z_j)}{(1 - \bar{z}_j z)}. \end{aligned} \tag{3.1}$$

Such  $P_i$ 's are analytic on the unit disk for  $|z_j| < 1$  and serve to eliminate poles of  $F_i$  at each  $z = z_j$  when formed into the products  $P_i(z)F_i(z)$ . Most importantly, each  $P_i$  is unimodular on the unit circle, and therefore we may replace  $F_i$  with  $P_i F_i$  in our bound Eq. (2.14) without changing the result. Since now both  $P_i F_i$  and  $\phi_i$  are analytic on the unit disc, Taylor expanding  $\phi_i P_i F_i$  about  $z = 0$  gives

$$F_i(z) = \frac{1}{P_i(z)\phi_i(z)} \sum_{n=0}^{\infty} a_n z^n. \tag{3.2}$$

Substituting this expression into Eq. (2.14) gives the central result

$$\sum_{n=0}^{\infty} |a_n|^2 \leq 1. \tag{3.3}$$

The coefficients  $a_n$  are different for each form factor, and must be determined by experiment. However, since both  $B$ - $D^*$  and  $B$ - $D$  states contribute to the same vector-vector dispersion relation, the sum of the squared-coefficient sums for  $f_+$  and  $g$  is bounded by one:

$$\sum_{n=0}^{\infty} (|a_n^{(f_+)}|^2 + |a_n^{(g)}|^2) \leq 1. \quad (3.4)$$

This relation holds if  $z$  in Eq. (3.2) is defined using  $m = m_{D^*}$  for  $g$  and  $m = m_D$  for  $f_+$ . An analogous result constrains the coefficients  $a_n$  of the form factors  $f_0$  and  $F_1$ ,

$$\sum_{n=0}^{\infty} (|a_n^{(f_0)}|^2 + |a_n^{(F_1)}|^2) \leq 1. \quad (3.5)$$

For the remainder of this paper, we content ourselves with the weaker constraint Eq. (3.3).

The utility of this parametrization arises from the observation that the physical range  $q_{\max}^2 \geq q^2 \geq 0$  for  $\bar{B} \rightarrow D^* l \bar{\nu}$  ( $D l \bar{\nu}$ ) semileptonic decays corresponds to  $0 \leq z \leq z_{\max} = 0.056(0.065)$ . We define an approximation  $F_i^N$  to the form factor  $F_i$  by truncating after the  $N$ th term:

$$F_i^N(z) = \frac{1}{P_i(z)\phi_i(z)} \sum_{n=0}^N a_n z^n. \quad (3.6)$$

Then the maximum error incurred by truncating after  $N$  terms is just

$$\begin{aligned} \max |F_i(z) - F_i^N(z)| &= \frac{1}{|P_i(z)\phi_i(z)|} \sum_{n=N+1}^{\infty} |a_n| z^n \\ &\leq \frac{1}{|P_i(z)\phi_i(z)|} \sqrt{\sum_{n=N+1}^{\infty} |a_n|^2} \sqrt{\sum_{n=N+1}^{\infty} z^{2n}} \\ &< \frac{1}{|P_i(z_{\max})\phi_i(z_{\max})|} \frac{z_{\max}^{N+1}}{\sqrt{1 - z_{\max}^2}}, \end{aligned} \quad (3.7)$$

where we have used the Schwarz inequality, Eq. (3.3), and the fact that  $z^{N+1}/|P_i(z)\phi_i(z)|$  increases monotonically over the physical range. For  $N$  as small as 2, this truncation error is quite small; see Table 2.

To calculate a relative error we need to estimate the form factor itself. This can be done at  $z = 0$  using heavy quark symmetries. The resulting bound on the relative error,  $|F_i(z) - F_i^N(z)|/F_i(0)$ , is shown in Table 2.

The larger relative error associated with  $F_1$  arises from a collusion of factors. Compared to  $f_0$ , these consist of a smaller value of  $\kappa$  and a greater value of  $s + p$ , both of which



decrease  $\phi_1$ , as well as a smaller normalization  $F_1(0)$ . The accuracy of the parametrization of  $F_1$  is improved to 0.34% by truncating after one more parameter (*i.e.*, taking  $N = 3$  above).

$i$	$F_i$	$ F_i(z) - F_i^N(z)  \times 10^2$	$ F_i(z) - F_i^N(z) /F_i(0)$
0	$f_0$	1.2	1.0%
1	$F_1$	4.6	6.1%
2	$g$	0.8	0.5%
3	$f_+$	1.4	1.3%

Table 2. *Bounds on truncation errors,  $|F_i(z) - F_i^N(z)|$ , for  $N = 2$ , for various form factors from Eq. (3.7). To estimate a corresponding relative error, we use the value of the form factor at threshold,  $F_i(0)$ , as predicted by heavy quark symmetries.*

#### 4. Branch Cuts

In the previous section we ignored branch cuts in the form factors with branch points inside  $|z| = 1$ . These cuts originate from non-resonant contributions with invariant masses below  $M + m$ . For example, branch points are expected at  $q^2 = (m_{B_c^*} + n m_\pi)^2$ , with  $n$  a positive integer, and at  $q^2 = (m_{\eta_{bc}} + m_\rho)^2$ , where  $\eta_{bc}$  is the pseudoscalar partner of the vector  $B_c^*$ . We now show that their neglect is quite justified.

We content ourselves with estimating the effect of any single cut modeled in a reasonable way, since multiple cuts can be handled analogously, and cuts modeled differently give comparable results.<sup>†</sup>

Any form factor  $g(q^2)$  in Eq. (2.1) satisfies a simple dispersion relation

$$g(q^2) = \frac{1}{\pi} \int_0^\infty dt \frac{\text{Im } g(t)}{t - q^2}. \quad (4.1)$$

A reasonable model for a cut can be obtained, say, by taking an additive contribution to  $g$  satisfying

$$\text{Im } g(t) = C \left( \sqrt{t - M_b^2} \theta(t - M_b^2) - \sqrt{t - M_a^2} \theta(t - M_a^2) \right), \quad (4.2)$$

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<sup>†</sup> The statement in Ref. [7] that the effect of such cuts may be incorporated by mapping them onto the unit circle and expanding in a new basis is erroneous; the matching of coefficients in the new basis to a Taylor expansion about  $z = 0$  involves an infinite number of equally important terms.

where  $q^2 = M_a^2$  is the location of the branch point of interest, and  $M_b$  is an arbitrary scale with  $M_b^2 > M_a^2$ . The subtraction is performed to ensure that  $\text{Im} g(t)$  vanishes as  $t \rightarrow \infty$ . The precise form of the subtraction is immaterial, so we choose one that simplifies our calculations. Moreover, since branch points on  $|z| = 1$  are irrelevant, we need consider only the case  $M_b < M + m$ . In all cases  $M_a > m_{B_c^*} + m_\pi$ , the location of the lowest branch point, which has  $z = -0.32$  for  $m = m_{D^*}$ .

The coefficient  $C$  arises as a coupling in diagrams connecting the  $(V - A)$  current to an external  $B$ - $D$  or  $B$ - $D^*$  pair through non-resonant on-shell intermediate states. The intermediate states couple to the current with a strength  $\hat{f}$ , and to  $B$ - $D$  or  $B$ - $D^*$  with strength  $\hat{g}$ . Furthermore, two-particle phase space provides a factor of  $1/8\pi$ . Phenomenologically  $C \approx \hat{f}\hat{g}/8\pi$  is expected to be quite small. We consider the most extreme case, namely,  $C = M^{s-3}c$  where  $c$  is dimensionless and at most of order unity.

Writing our model cut from Eq. (4.2) in terms of the variable  $z$ , we have

$$g_{\text{cut}}(z) = 4cM^{s-2}\sqrt{r} \left( \frac{\sqrt{(z-z_a)(1-zz_a)}}{(1-z)(1-z_a)} - \frac{\sqrt{(z-z_b)(1-zz_b)}}{(1-z)(1-z_b)} \right). \quad (4.3)$$

Let  $f(z)$  stand for any of the form factors, with corresponding functions  $\phi(z)$  from Eq. (2.15) and  $P(z)$  from Eq. (3.1). Consider the difference  $\tilde{f}(z) = f(z) - g_{\text{cut}}(z)$ , and let  $f_{\text{cut}} = g_{\text{cut}}\phi P$ . The function  $\tilde{f}\phi P$  is thus designed to be analytic on the unit disc. We proceed in two steps: First we find a bound for  $\tilde{f}\phi$  analogous to that in (2.14); this constraint translates into a new bound on the parameters in our expansion. Then we show that  $f_{\text{cut}}$  is well approximated in the physical region by a polynomial of low degree, so that truncating our expansion after only a few terms incurs a very small error.

By the Minkowski inequality and (2.14) we have

$$\left( \int_0^{2\pi} d\theta |\tilde{f}\phi|^2 \right)^{1/2} \leq \left( \int_0^{2\pi} d\theta |f\phi|^2 \right)^{1/2} + \left( \int_0^{2\pi} d\theta |f_{\text{cut}}|^2 \right)^{1/2} \leq \sqrt{2\pi}(1 + I_{\text{cut}}^{1/2}), \quad (4.4)$$

where  $z = e^{i\theta}$  and

$$I_{\text{cut}} \equiv \frac{1}{2\pi} \int_0^{2\pi} d\theta |f_{\text{cut}}|^2. \quad (4.5)$$

As before, the functions  $P(z)$  are unimodular on the unit circle, and so leave the integrals unchanged.  $I_{\text{cut}}$  can be computed using the explicit form for the cut in Eq. (4.3). As a specific example, take for  $\phi$  in Eq. (2.15) the case of  $p = 3$  and  $s = 1$ , corresponding to

the form factor  $g$ . The numbers to follow are specific to the case  $m = m_{D^*}$ , although the qualitative results are the same for  $m = m_D$ . For  $z_a = -0.32$  and  $z_b = -1.0$ , we find

$$I_{\text{cut}} \approx 2.2 \times 10^{-3} c^2. \quad (4.6)$$

Thus the bound on  $\tilde{f}$  is relaxed only by  $\sim 5\%$  times  $c$  relative to that on  $f$ . A realistic choice of coefficient  $c$  significantly improves this bound, as does a branch point closer to the  $B$ - $D$  or  $B$ - $D^*$  threshold  $z = -1$ . For example, for  $z_a = -0.5$  replace  $2.2 \times 10^{-3}$  by  $4.0 \times 10^{-4}$  in Eq. (4.6).

Being analytic,  $\tilde{f}\phi P$  has a Taylor expansion for  $|z| \leq 1$ . Hence one may write

$$f(z) = \frac{1}{\phi(z)P(z)} \left( \sum_{n=0}^{\infty} a_n z^n + f_{\text{cut}} \right). \quad (4.7)$$

Here the coefficients  $a_n$  are bounded,  $\sum_{n=0}^{\infty} |a_n|^2 \leq (1 + I_{\text{cut}}^{1/2})^2$ . Moreover,  $f_{\text{cut}}$  is analytic over the physical region. Define the remainder  $R_{\text{cut}}^N$  through

$$f_{\text{cut}}(z) = \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^N b_n z^n + R_{\text{cut}}^N(z). \quad (4.8)$$

The remainder  $R_{\text{cut}}^N$  is the additional error introduced in the parametrization of  $f(z)$  as an  $N$ -th order polynomial in  $z$  with coefficients  $a_n + b_n$ . These coefficients obey a constraint almost identical to Eq. (3.3), because the  $b_n/c$  are uniformly small:  $\sum_{n=0}^{n=N} |a_n + b_n|^2 \leq (1 + I_{\text{cut}}^{1/2})^2 (1 + \sum_{n=0}^{n=N} |b_n|)^2 = (1.07)^2$ ,  $(1.07)^2$ , and  $(1.08)^2$ , for  $c = 1$  and  $N = 2, 3$ , and  $4$ , respectively. The maximum of  $R_{\text{cut}}^N(z)$  over the physical region is  $-1 \times 10^{-6}$ ,  $2 \times 10^{-7}$ , and  $4 \times 10^{-9}$  times  $c$  for  $N = 2, 3$ , and  $4$ , respectively. These figures should be compared with the bound on the truncation in the analytic part,  $\sum_{n=N+1}^{\infty} a_n z^n \leq (1 + I_{\text{cut}}^{1/2})(0.056)^{N+1} \approx 2 \times 10^{-4}$ ,  $1 \times 10^{-5}$ , and  $6 \times 10^{-7}$ . Since we expect realistic branch cuts to have  $c \ll 1$ , we see that their effect is negligible.

## 5. Uncertainties

Two important statements follow immediately from Eqs. (3.2) and (3.3):

- a ) Each of the various  $\bar{B} \rightarrow D$  or  $\bar{B} \rightarrow D^*$  form factors can be accurately described by three parameters, one of which is the normalization at zero recoil, with a truncation error of order 1%;

b ) The fitting parameters obey  $\sum_{n=0}^{n=2} |a_n|^2 \leq B^2$ , with  $B = 1$ .

A number of approximations have been made in deriving these results. How do corrections to these approximations alter the above statements? We answer this question by noting that nearly all the corrections we expect to be non-negligible can be taken into account by altering the bound to  $B \neq 1$ .

An estimate of how much  $B$  might change may be made by considering uncertainties arising from the following sources:

- 1 ) The  $b$  and  $c$  quark masses, which enter into the one-loop perturbative functions  $\chi_J$ , are not well established; we take  $m_c/m_b = 0.33$ . This leads to a roughly 5% uncertainty in the normalization of  $\phi$  which, by redefining the  $a_n$ , is equivalent to a 5% uncertainty in the value  $B$  bounding the  $a_n$ .
- 2 ) The functions  $\chi_J$  also receive perturbative two-loop corrections. Since  $\phi$ , and therefore the bound, depends only on  $(\chi_J)^{1/2}$ , such corrections should lead to no more than a 15% change in  $B$ .
- 3 ) The masses of the  $B_c^*$  poles were computed from a potential model. Computations by various groups typically agree to a fraction of a percent[11–13]. The results from two different groups[11,12] give Blaschke factors that agree to 2%. However,  $P(z)$  is sensitive to the mass of the  $3S$  vector state, which is close to the  $B$ - $D^*$  threshold, and is presented only by [12]. Changing it by 1% results in a 20% change in  $P(0)$ ;  $P(z)/P(0)$ , however, varies by less than 1%.
- 4 ) We argued in the previous section that contributions from multi-particle cuts should alter the bound  $B$  by less than 8%.
- 5 ) In extracting values of  $|V_{cb}|\mathcal{F}(1)$ , we require bounds not on  $a_1$  and  $a_2$  alone, but on  $a_1/\mathcal{F}(1)$  and  $a_2/\mathcal{F}(1)$ , and these depend on the zero-recoil normalization  $\mathcal{F}(1)$ . This normalization is predicted to no worse than 20% accuracy by heavy quark symmetry[8].

The uncertainties (1) to (5) are uncorrelated, so to estimate their total effect, we add them in quadrature. This leads to a relaxation on our bounds from  $B = 1$  to less than  $B = 1.4$ .

This relaxation of the bound increases the truncation error on, for example,  $f_0$  from 0.012 to 0.017, still negligibly small given the current experimental accuracy. Even if we added the uncertainties linearly, the truncation error would only rise to 0.020. We see that statement (a) is extraordinarily robust; it is nearly independent of the size of the uncertainties listed above.

On the other hand, allowing a value of  $B$  larger than 1 in statement (b) could in principle affect the extraction of  $|V_{cb}|$  significantly due to a larger allowed range for  $a_1$  and  $a_2$ . For this reason, it would be useful to pin down the  $\alpha_s$  corrections and the mass of the  $3S$  vector  $B_c^*$  state more precisely. However, for the extraction we perform in Sec. 7, relaxing the bound from  $B = 1$  to 1.4 turns out to change the results very slightly: The central values of  $|V_{cb}|\mathcal{F}(1)$  and the slope change by no more than a tenth of a standard deviation.

One should also bear in mind that our bounds can be significantly improved by the inclusion of more terms than just  $B$ - $D^*$  or  $B$ - $D$  pairs to saturate the bound in Eq. (2.8). Such contributions arise through higher resonances of the current  $J$ ; if estimated numerically, they have an effect equivalent to reducing  $B$ .

## 6. Heavy Quark Symmetry

The parametrizations Eq. (3.2) make no use of heavy quark symmetry. Thus,  $1/m_c$  corrections to the extraction of  $|V_{cb}|$  from  $\bar{B} \rightarrow Dl\bar{\nu}$  decays enter only through the normalization of the form factor  $f_+(z=0)$  at zero recoil. This normalization is determined by heavy quark symmetry to  $\mathcal{O}(1/m_c)$ .

If the individual  $\bar{B} \rightarrow D^*l\bar{\nu}$  form factors  $f_0$ ,  $F_1$ , and  $g$  are experimentally determined in the near future, separate extractions of  $|V_{cb}|$  can be made for each form factor. These extractions will depend on heavy quark symmetry only through the normalization of form factors at zero recoil. This is useful because the normalization of  $f_0$  is predicted to  $\mathcal{O}(1/m_c^2)$ [15].

At present, to extract  $|V_{cb}|$  from the  $\bar{B} \rightarrow D^*l\bar{\nu}$  differential width (2.3) in terms of our three-parameter descriptions, one must relate  $f_0$ ,  $a_+$ , and  $g$  using heavy quark symmetry. In the infinite mass limit, all form factors for  $\bar{B} \rightarrow D$  and  $\bar{B} \rightarrow D^*$  (as well as  $B \rightarrow B$ ) are directly proportional to the universal Isgur-Wise function[1]. Consequently, the ratio of any two form factors assumes a simple form:

$$a_+/g = -\frac{1}{2}, \quad f_0/a_+ = -2M^2r(\omega+1), \quad \text{and} \quad f_0/g = M^2r(\omega+1), \quad (6.1)$$

where  $\omega = v \cdot v'$  is the product of the  $\bar{B}$  with  $D$  or  $D^*$  meson velocities. These ratios admit two types of correction, namely power corrections in  $1/m_c$ , and running and matching corrections relating QCD to the heavy quark effective theory. We discuss each of these in turn.

Because many heavy quark symmetry-violating contributions cancel in the above ratios, one might expect  $1/m_c$  corrections to be smaller than in, say, the relation between the  $B \rightarrow B$  elastic and  $\bar{B} \rightarrow D$  or  $\bar{B} \rightarrow D^*$  semileptonic form factors. For example,  $a_+/g = -\frac{1}{2}$  may be derived using only charm quark spin symmetry, without recourse to bottom-charm flavor symmetry; spin symmetry is expected to hold more precisely than the full flavor-spin symmetry[16]. In addition, the ratio  $f_0/g = M^2 r(\omega + 1)[1 + (\omega - 2)\bar{\Lambda}/2m_c]$  involves no unknown  $\omega$ -dependent functions[15] at  $\mathcal{O}(1/m_c)$ , but only the constant  $\bar{\Lambda} = m_D - m_c$ . The third ratio is given by the quotient of these two. Choosing two different pairs of the above ratios gives two different parametrizations of the decay form factor  $\mathcal{F}(\omega)$  conventionally defined by

$$\frac{d\Gamma}{d\omega} = \frac{|V_{cb}|^2 G_F^2}{48\pi^3} m^3 (M - m)^2 \sqrt{\omega^2 - 1} \left[ 4\omega(\omega + 1) \frac{1 - 2\omega r + r^2}{(1 - r)^2} + (\omega + 1)^2 \right] \mathcal{F}^2(\omega). \quad (6.2)$$

In the heavy quark limit, the form factor  $\mathcal{F}(\omega)$  is simply the Isgur-Wise function times QCD corrections (discussed below), and we readily see that Eq. (2.3) reduces to Eq. (6.2). In terms of the parametrizations Eq. (3.2) of  $g$  and  $f_0$ , respectively,

$$\mathcal{F}(z) = \frac{1}{P_2(z)\phi_2(z)} [P_2(0)\phi_2(0)\mathcal{F}(z=0) + M\sqrt{r}(a_1 z + a_2 z^2)] \quad (6.3a)$$

$$= \frac{(1 - z)^2}{P_0(z)\phi_0(z)(1 + z)^2} [P_0(0)\phi_0(0)\mathcal{F}(z=0) + \frac{1}{2M\sqrt{r}}(b_1 z + b_2 z^2)]. \quad (6.3b)$$

The form factor may be expressed as a function  $\mathcal{F}(\omega)$  of velocity transfer by rewriting  $z$  as

$$z = \frac{\sqrt{\frac{\omega+1}{2}} - 1}{\sqrt{\frac{\omega+1}{2}} + 1}. \quad (6.4)$$

At zero recoil,  $\mathcal{F}(\omega = 1) = 1$  times corrections whose estimates range from 0.89 to 0.99[8]. Relative to this normalization, the parametrization in Eq. (6.3a) has a 0.5% truncation error, while that in Eq. (6.3b) has a 1.0% truncation error; see Table 2.

To the degree that  $1/m_c$  corrections are negligible, the extracted values of  $|V_{cb}|\mathcal{F}(\omega=1)$  and the slope  $\mathcal{F}'(\omega = 1)$  cannot depend on which of the parametrizations (6.3) we use. Since  $1/m_c$  corrections enter differently into each of these parametrizations, the degree to which this is true gauges the sensitivity of the extraction to heavy quark symmetry violations.

For a thorough accounting of relations between form factors when using heavy quark symmetry, one must also include effects due to the running of the QCD coupling  $\alpha_s$  and

matching between the full theory of QCD and the heavy quark effective theory. The form factors are then no longer just trivial factors times the Isgur-Wise function, but now include a functional dependence on  $\omega$ , as well as  $m_c$ ,  $m_b$ , and the value of  $\alpha_s$  at these scales. For conciseness and definiteness, we adopt the notation of Neubert[17] to parametrize such corrections. The relation between the Isgur-Wise function  $\xi(\omega)$  and the relevant form factors then reads

$$a_+ = -\frac{1}{2M\sqrt{r}}(\hat{C}_1^5 + \hat{C}_2^5 r + \hat{C}_3^5)\xi, \quad f_0 = M\sqrt{r}(\omega + 1)\hat{C}_1^5\xi, \quad g = \frac{1}{M\sqrt{r}}\hat{C}_1\xi. \quad (6.5)$$

The functions  $\hat{C}_1$ ,  $\hat{C}_1^5$  become unity when the strong coupling is switched off, whereas the other  $\hat{C}$ 's vanish. In this limit we recover the ratios in Eq. (6.1).

Apart from changing the overall normalization of form factors at zero recoil by a few percent, the functional dependences in Eq. (6.5) turn out to be rather weak over the allowed range for  $\bar{B} \rightarrow D$  or  $\bar{B} \rightarrow D^*$  semileptonic decay ( $\omega = 1.0$  to  $1.5$ ). In particular,  $\hat{C}_1$  decreases from 1.136 to 1.011 over this range, but  $-2a_+/g = 0.864 \rightarrow 0.882$ , and  $f_0/gM^2r(\omega + 1) = 0.868 \rightarrow 0.884$  are nearly constant. In addition, corrections due to running between the bottom and charm mass scales cancel out of such ratios.

Because the undetermined  $1/m_c$  corrections are just as significant, there is little to be gained in incorporating the calculated matching corrections explicitly in our analysis; rather, our sensitivity to both  $1/m_c$  and matching corrections is gauged by comparing the extractions of  $|V_{cb}|\mathcal{F}(1)$  and the slope  $\mathcal{F}'(1)$  by the two parametrizations of Eq. (6.3). Compared to the  $g$  parametrization, the  $f_0$  parametrization changes the central values of CLEO's  $|V_{cb}|\mathcal{F}(1)$ , and both CLEO's and ARGUS's  $\mathcal{F}'(1)$  by less than a fourth of a standard deviation; ARGUS's and ALEPH's  $|V_{cb}|\mathcal{F}(1)$ , as well as ALEPH's  $\mathcal{F}'(1)$ , change by less than a tenth of a standard deviation (*i.e.*,  $< 2\%$  for all  $|V_{cb}|\mathcal{F}(1)$ ).

## 7. Results

Since both parametrizations (6.3) give essentially the same results, we choose the  $g$  parametrization, which has a smaller truncation error. From the point of view of heavy quark symmetry, one should use the  $f_0$  parametrization, since  $f_0(\omega = 1)$  is predicted to higher accuracy. Here we are more concerned with exploring the implications of our parametrizations. The central values and 68% confidence intervals should be taken as indicative; proper inclusion of efficiencies, resolutions, and correlated errors can only be done by the experimental groups themselves.

Fitting  $|V_{cb}|\mathcal{F}(1)$ ,  $a_1/\mathcal{F}(1)$ , and  $a_2/\mathcal{F}(1)$  to experiment yields the results in Table 3. For each experiment, we have listed the best fit values for  $|V_{cb}|\mathcal{F}(1)$ ,  $a_1/\mathcal{F}(1)$ , and  $a_2/\mathcal{F}(1)$ , as well as the resulting slope  $\mathcal{F}'(\omega = 1)$ . The 68% confidence intervals due to statistics are included as well. The parametrization (6.3a) includes the constraint  $\sum_{n=0}^{\infty} |a_n|^2 \leq 1$ ; for comparison, we also present the best fit values resulting from an unconstrained fit with freely varying  $a_n$ .

$B$	$ V_{cb} \mathcal{F}(1) \cdot 10^3$	$a_1/\mathcal{F}(1)$	$a_2/\mathcal{F}(1)$	$\mathcal{F}'(1)$	Expt.
1	$35.7^{+3.7}_{-2.8}$	$0.046^{+0.05}_{-0.14}$	$-1.00^{+2.0}_{-0.0}$	$-0.89^{+0.3}_{-0.8}$	CLEO
$\infty$	$33.3^{+6.1}_{-6.1}$	$0.181^{+0.38}_{-0.27}$	$-3.20^{+4.5}_{-5.9}$	$-0.14^{+2.1}_{-1.5}$	CLEO
1	$45.8^{+8.1}_{-10.9}$	$-0.200^{+0.22}_{-0.07}$	$0.98^{+0.0}_{-2.0}$	$-2.3^{+1.2}_{-0.4}$	ARGUS
$\infty$	$49.5^{+19.4}_{-19.5}$	$-0.297^{+0.71}_{-0.32}$	$2.59^{+5.4}_{-11.3}$	$-2.8^{+3.9}_{-1.8}$	ARGUS
1	$31.5^{+4.5}_{-5.8}$	$0.090^{+0.25}_{-0.10}$	$1.00^{+0.0}_{-2.0}$	$-0.65^{+1.4}_{-0.6}$	ALEPH
$\infty$	$31.8^{+7.5}_{-7.5}$	$0.073^{+0.52}_{-0.33}$	$1.33^{+5.3}_{-7.8}$	$-0.74^{+2.9}_{-1.8}$	ALEPH

Table 3. *Fit values for  $|V_{cb}|\mathcal{F}(1)$ ,  $a_1/\mathcal{F}(1)$ ,  $a_2/\mathcal{F}(1)$ , and the zero recoil slope of  $\mathcal{F}(\omega)$  from the various experiments, constrained to obey  $\sum_{n=0}^{n=2} |a_n|^2 \leq B$ .*

The fits allowed by QCD are those with (in particular)  $|a_2| \leq 1$ . The extracted values of  $|V_{cb}|$  are in good agreement with a previous extraction[7], after accounting for differences in definitions and experimental data. We have renormalized the ARGUS data to bring their assumed  $B$  lifetime and  $D^0 \rightarrow K^- \pi^+$  branching ratio into agreement with more recent experiments; we use  $\tau_B = 1.61$  psec and [18]  $B(D^0 \rightarrow K^- \pi^+) = 4.01\%$ .

The central values for  $|V_{cb}|\mathcal{F}(1)$  agree surprisingly closely with those of the experimental groups themselves. This did not need to be the case, as one can see from the behavior of the unconstrained fit.

The connection between our parameters  $a_1, a_2$  and the commonly used expansion in  $(\omega - 1)$  is

$$\begin{aligned}
\frac{\mathcal{F}(\omega)}{\mathcal{F}(1)} = & 1 + \left[ 5.54 \frac{a_1}{\mathcal{F}(1)} - 1.15 \right] (\omega - 1) + \left[ -7.73 \frac{a_1}{\mathcal{F}(1)} + 0.69 \frac{a_2}{\mathcal{F}(1)} + 1.11 \right] (\omega - 1)^2 \\
& + \left[ 8.19 \frac{a_1}{\mathcal{F}(1)} - 1.14 \frac{a_2}{\mathcal{F}(1)} - 0.99 \right] (\omega - 1)^3 + \dots
\end{aligned}
\tag{7.1}$$

While such an expansion describes the form factor well close to zero recoil, it converges poorly over the rest of the kinematic range. Substituting the allowed range of parameters  $\sum_{n=0}^{n=2} |a_n|^2 \leq 1$  gives a truncation error for a quadratic fit in  $(\omega - 1)$  of 120%; the truncation



error of a linear fit is 220%. To be assured of fitting a QCD-allowed form factor at percent-level accuracy, a parametrization obeying the same constraints as Eq. (6.3) must be used.

Plotted in Fig. 1 are the constrained and unconstrained fits to the CLEO[5] data. Both fits match the data well; the chi-squares per degree of freedom are  $\chi^2/dof = 0.65$  and 0.50, respectively. The CLEO group extracts  $|V_{cb}|\mathcal{F}(1) \cdot 10^3 = 35.1 \pm 1.9(stat)$  and a slope  $\mathcal{F}'(1) = -0.84 \pm 0.13$  using a linear fit, in close agreement with our bounded fit. The unbounded fit serves as an illustration of a parametrization which gives a markedly different best fit; the central value of  $|V_{cb}|\mathcal{F}(1)$  differs by 5% from the linear result, while the slope is in violation of the Bjorken bound[19],  $\mathcal{F}'(1) < -1/4$ . By Eq. (3.3), this unconstrained fit is ruled out by QCD.

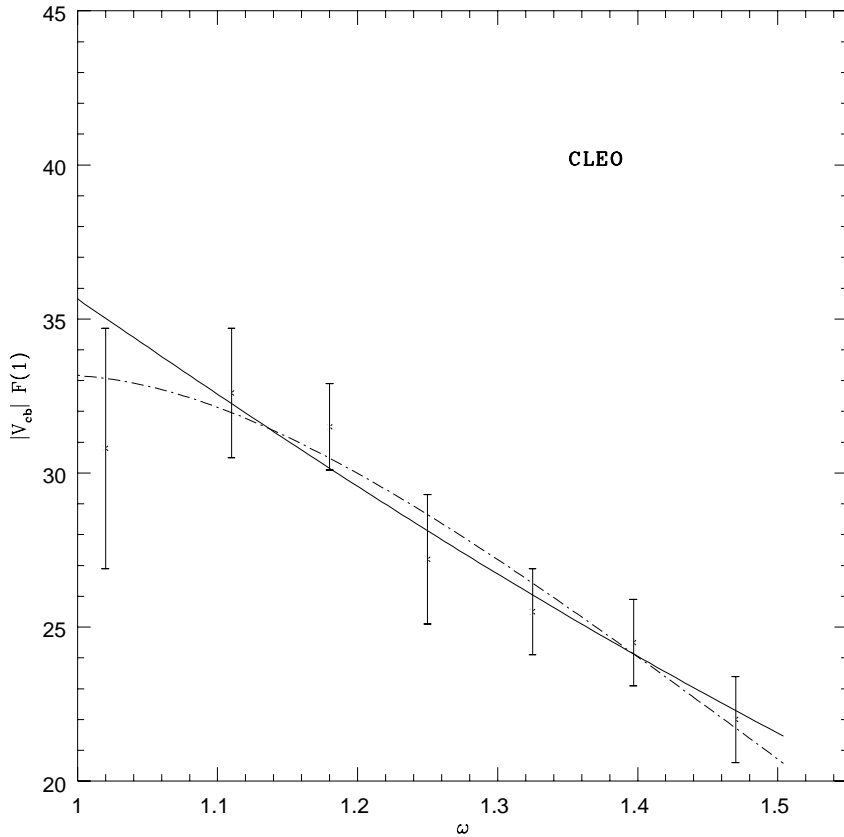


Figure 1. *Fit of CLEO data to our parametrization, Eq. (6.3a). The solid line shows the result of imposing the QCD-derived constraint  $\sum_{n=0}^{n=2} |a_n|^2 \leq 1$  on the parametrization. The dot-dash line shows the corresponding unconstrained fit.*

For ALEPH[4] the constrained and unconstrained fits overlay each other quite closely (Fig. 2). A linear fit by the ALEPH group gives  $|V_{cb}|\mathcal{F}(1) \cdot 10^3 = 31.4 \pm 2.3(stat)$  and a slope  $\mathcal{F}'(1) = -0.39 \pm 0.21$ , in good agreement with the results of our constrained fit. The

confidence intervals in Table 3 are somewhat larger for ALEPH than might be expected because of the smallness of the minimum  $\chi^2$ : Both the bounded and unbounded fits have  $\chi^2/dof = 0.37$ , so a larger range of fit parameters fall within the 68% confidence limits in either case.

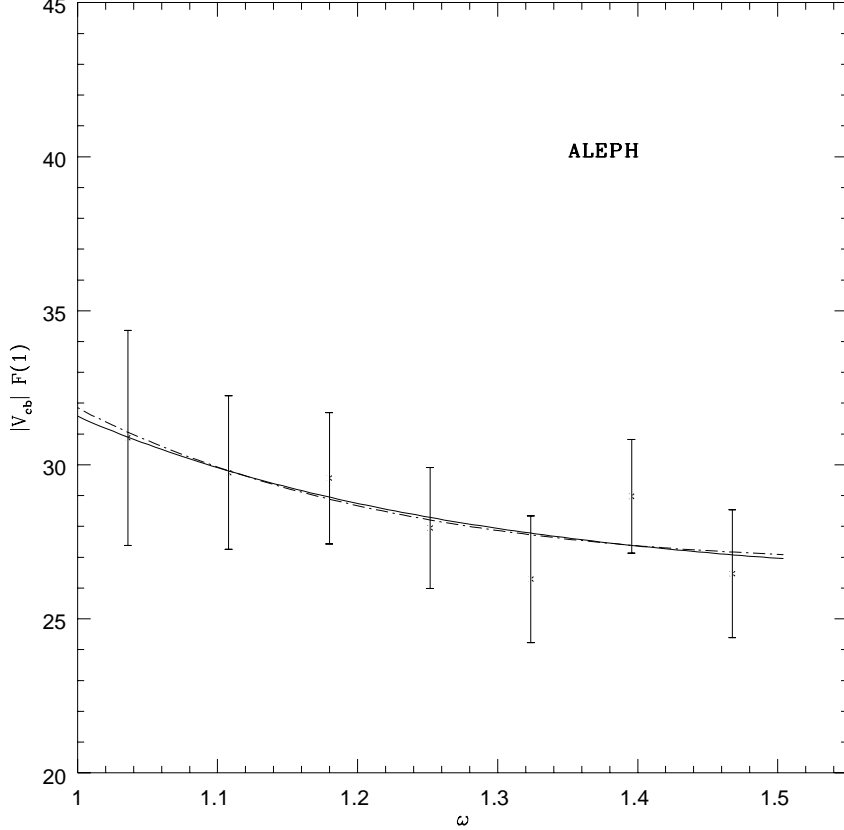


Figure 2. *Fit of ALEPH data to our parametrization; see Figure 1 caption for details.*

The constrained and unconstrained fits to ARGUS[6] data differ mainly near zero recoil, with comparable values  $\chi^2/dof = 0.70$  and  $0.67$ , respectively (Fig. 3). The ARGUS group used several parametrizations, which yielded central values of  $|V_{cb}|\mathcal{F}(1) \cdot 10^3$  from 39 to 46. Their linear fit gave  $|V_{cb}|\mathcal{F}(1) \cdot 10^3 = 39 \pm 4$  and  $\mathcal{F}'(1) = -1.17 \pm 0.11$ , in some contrast to our central values.

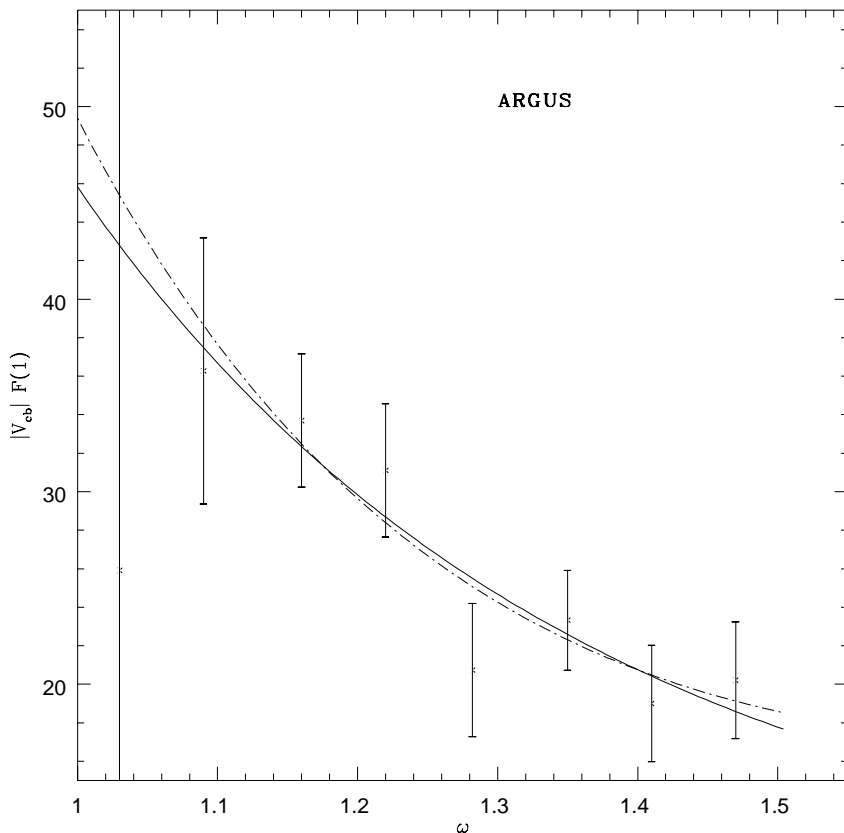


Figure 3. *Fit of ARGUS data to our parametrization; see Figure 1 caption for details.*

Although the large statistical uncertainty in  $a_2/\mathcal{F}(1)$  precludes its determination at present, we can make a definite prediction for the future: The central value of  $a_2/\mathcal{F}(1)$  must increase from CLEO's present (unconstrained fit) number to fall inside our bounds. Taking the theoretical estimates of Sec. 5 into account, we predict  $|a_2/\mathcal{F}(1)| \leq 1.4$ .

## 8. Conclusions

Dispersion relation techniques and the use of analyticity properties of hadronic form factors as functions of their kinematic variables provide a valuable window into the realm of nonperturbative physics. Using these methods, one can obtain useful bounds on quantities of interest, in this case the form factors in the semileptonic decays  $\bar{B} \rightarrow Dl\bar{\nu}$  and  $\bar{B} \rightarrow D^*l\bar{\nu}$ .

These bounds may be transformed into parametrizations of the four experimentally accessible form factors relevant to  $\bar{B} \rightarrow Dl\bar{\nu}$  and  $\bar{B} \rightarrow D^*l\bar{\nu}$ . Given the continuing experimental scrutiny devoted to these decays, these form factors will likely be measured in the foreseeable future.

Our derivation of these parametrizations relied on dispersion relations, crossing symmetry, and a perturbative QCD calculation performed at a scale  $m_B + m_{D^*}$ . The derivation improves on an earlier work[7] in that no use of heavy quark symmetry was made. The various uncertainties involved in the derivation, such as perturbative corrections and uncertainties in quark masses, were estimated, and shown to be unimportant. This includes effects from branch cuts in the form factors due to non-resonant contributions.

The result is a three-parameter description of each of the form factors  $f_0$ ,  $g$ , and  $f_+$  accurate over the entire physical kinematic range to better than 2%. The value of one of the parameters, the normalization at zero recoil  $\mathcal{F}(1)$ , is predicted by heavy quark symmetry. The other two parameters  $a_1, a_2$  are constrained by  $|a_1|^2 + |a_2|^2 \leq B^2$ , with a leading-order result  $B = 1$ . A very conservative estimate of corrections to our results leads us to conclude that to all orders, the bound obeys  $B < 1.4$ . The 2% or better accuracy of the three-parameter description applies for any  $B \leq 1.4$ . The three parameter fit to the form factor  $F_1$  is less accurate; for  $B \leq 1.4$  we find a bound of 8% on its relative error.

We emphasize that we have determined strict upper bounds on the truncation errors. The truncation errors may be significantly smaller. The strict inequality (2.8) can be improved by including the contributions of other intermediate states; our use of Blaschke factors, Eq. (3.1), amounts to assuming the largest possible uncertainty from the residues of poles in the form factors; and the bounds on the parameters for each form factor, Eq. (3.3), are actually correlated, as in Eqs. (3.4) and (3.5).

As an application of our results, the individual form factors in  $\bar{B} \rightarrow D^* l \bar{\nu}$  were combined using heavy quark symmetries in order to obtain a single parametrization of the differential cross-section  $d\Gamma/dq^2$ , which was then fit to data. This was necessary because the best data currently available sums over  $D^*$  mesons in all polarization states and thus involves more than one form factor. However, to  $\mathcal{O}(1/m_c)$ , our results depend only on charm quark spin symmetry and the constant  $\bar{\Lambda} = m_D - m_c$ , and are therefore expected to be more reliable than those using the full flavor-spin heavy quark symmetry. We obtain values for a three-parameter ( $|V_{cb}|\mathcal{F}(1)$ ,  $a_1/\mathcal{F}(1)$ , and  $a_2/\mathcal{F}(1)$ ) fit to the single form factor  $\mathcal{F}(v \cdot v')$  describing  $\bar{B} \rightarrow D^* l \bar{\nu}$  that is free of the theoretical errors inherent in choosing a parametrization for extrapolating the data to zero recoil. We again emphasize that, although heavy quark spin symmetries were used in obtaining values for  $|V_{cb}|\mathcal{F}(1)$ , this is a limitation imposed by the currently available data that will be lifted when better measurements of the individual form factors become available.

The intensive experimental effort focused on semileptonic  $\bar{B} \rightarrow Dl\bar{\nu}$  and  $\bar{B} \rightarrow D^*l\bar{\nu}$  decays will result in increasingly precise measurements of the rate and form factors. Our descriptions of these form factors are remarkably insensitive to theoretical uncertainties, and are accurate over the physical kinematic range to better than 2%; as such, they should be useful ingredients in studying the nonperturbative physics of semileptonic  $B$  decays.

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## References

- [1] N. Isgur and M.B. Wise, Phys. Lett. B **232** (1989) 113 and Phys. Lett. B **237** (1990) 527.
- [2] E. Eichten and B. Hill, Phys. Lett. B **234** (1990) 511.
- [3] M. B. Voloshin and M. A. Shifman, Yad. Fiz. **47**, (1988) 801 [Sov. J. Nucl. Phys. **47** (1988) 511].
- [4] D. Buskulic *et al.* (ALEPH Collaboration), CERN Report No. CERN-PPE/95-094 (unpublished).
- [5] B. Barish *et al.* (CLEO Collaboration), Phys. Rev. **D51** (1995) 1014.
- [6] H. Albrecht *et al.* (ARGUS Collaboration), Z. Phys. C **57** (1993) 533.
- [7] C.G. Boyd, B. Grinstein, and R.F. Lebed, U.C. San Diego Report No. UCSD/PTH 95-03 [hep-ph/9504235], to appear in Phys. Lett. B.
- [8] A.F. Falk and M. Neubert, Phys. Rev. **D47** (1993) 2695 and 2982; M. Neubert, Phys. Lett. B **338** (1994) 84; T. Mannel, Phys. Rev. **D50** (1994) 428; M. Shifman, N. Uraltsev, and A. Vainshtein, Phys. Rev. **D51** (1995) 2217.
- [9] N.N. Meiman, Sov. Phys. JETP **17** (1963) 830; S. Okubo, Phys. Rev. **D3** (1971) 2807; S. Okubo and I. Fushih, Phys. Rev. **D4** (1971) 2020; V. Singh and A.K. Raina, Fortschritte der Physik **27** (1979) 561; C. Bourrely, B. Machet, and E. de Rafael, Nucl. Phys. **B189** (1981) 157; E. de Rafael and J. Taron, Phys. Rev. **D50** (1994) 373.
- [10] C.G. Boyd, B. Grinstein, and R.F. Lebed, Phys. Rev. Lett. **74** (1995) 4603.
- [11] V.V. Kisilev, A.K. Likhoded, and A.V. Tkabladze, Phys. Rev. **D51** (1995) 3613.
- [12] E.J. Eichten and C. Quigg, Phys. Rev. **D49** (1994) 5845.
- [13] Y.-Q. Chen and Y.-P. Kuang, Phys. Rev. **D46** (1992) 1165.
- [14] P. Duren, *Theory of  $H^p$  Spaces*, Academic Press, New York, 1970.
- [15] M.E. Luke, Phys. Lett. B **252** (1990) 447.
- [16] B. Grinstein, in *Proceedings of the Workshop on B Physics at Hadron Accelerators*, Snowmass, Colorado, 1993, ed. C.S. Mishra and P. McBride (Fermilab, 1994).
- [17] M. Neubert, Phys. Rep. **245** (1994) 259.
- [18] L. Montanet *et al.* (Particle Data Group), Phys. Rev. **D50** (1994) 1173.
- [19] J. Bjorken, in *Proceedings of the 4th Recontres de Physique de la Vallée d'Aoste*, La Thuille, Italy, 1990, ed. M. Greco (Editions Frontières, Gif-Sur-Yvette, France, 1990); N. Isgur and M.B. Wise, Phys. Rev. **D43** (1991) 819.